

# The Quotient of Two Independent Random Variables

BIO210 Biostatistics

Extra Reading Material for Lecture 29

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# 1 The General Case

Let  $V$  and  $W$  be two independent *random variables* with PDF  $f_V(v)$  and  $f_W(w)$ , respectively.

Now we define a new random variable  $Q = \frac{V}{W}$ . We want to compute the PDF of  $Q$ :

$$f_Q(q)$$

Using similar techniques when we derived the *convolution formula* in the **Extra Reading Material** from **Lecture 13**, we start with the CDF of  $Q$ . In addition, when  $W$  takes a value,  $Q$  is basically a linear function of  $V$ . This makes the derivation a lot easier.

We start with the definition:

$$F_{Q|W}(q|w) = \mathbb{P}(Q \leq q | w) = \mathbb{P}\left(\frac{V}{w} \leq q \mid w\right) \quad (1)$$

## 1.1 When $w > 0$

When  $w > 0$ , equation (1) becomes:

$$F_{Q|W}(q|w) = \mathbb{P}(V \leq wq | w) = F_{V|W}(wq | w) \quad (2)$$

Take the derivative at both sides with respect to  $Q$ , equation (2) becomes:

$$f_{Q|W}(q|w) = w \cdot f_{V|W}(wq | w) \quad (3)$$

By definition, we have the joint PDF of  $Q$  and  $W$ :

$$f_{Q,W}(q, w) = f_W(w) \cdot f_{Q|W}(q | w) \quad (4)$$

Putting equation (3) into equation (4), we have:

$$f_{Q,W}(q, w) = f_W(w) \cdot w \cdot f_{V|W}(wq | w)$$

Since  $V$  and  $W$  are independent, we can remove the conditioning:

$$f_{Q,W}(q, w) = w \cdot f_W(w) f_V(wq)$$

Integrate over  $\mathbf{W}$  under the condition  $w > 0$ , we get the marginal PDF of  $\mathbf{X}$ :

$$\mathbf{f}_{\mathbf{Q}}(q) = \int_0^{\infty} \mathbf{f}_{\mathbf{Q},\mathbf{W}}(q, w)dw = \int_0^{\infty} w \cdot \mathbf{f}_{\mathbf{W}}(w)\mathbf{f}_{\mathbf{V}}(wq)dw \quad (5)$$

## 1.2 When $\mathbf{W} < 0$

When  $w < 0$ , equation (1) becomes:

$$\mathbb{F}_{\mathbf{Q}|\mathbf{W}}(q | w) = \mathbb{P}(\mathbf{V} \geq wq | w) = 1 - \mathbb{F}_{\mathbf{V}|\mathbf{W}}(wq | w) \quad (6)$$

Take the derivate at both sides with respect to  $\mathbf{Q}$ , equation (6) becomes:

$$\mathbf{f}_{\mathbf{Q}|\mathbf{W}}(q | w) = -w \cdot \mathbf{f}_{\mathbf{V}|\mathbf{W}}(wq | w) \quad (7)$$

Similarly, put equation (7) into equation (4), we have:

$$\mathbf{f}_{\mathbf{Q},\mathbf{W}}(q, w) = \mathbf{f}_{\mathbf{W}}(w) \cdot (-w) \cdot \mathbf{f}_{\mathbf{V}|\mathbf{W}}(wq | w)$$

Again, since  $\mathbf{V}$  and  $\mathbf{W}$  are independent, we can remove the conditioning:

$$\mathbf{f}_{\mathbf{Q},\mathbf{W}}(q, w) = -w \cdot \mathbf{f}_{\mathbf{W}}(w)\mathbf{f}_{\mathbf{V}}(wq)$$

Integrate over  $\mathbf{W}$  under the condition  $w < 0$ , we get the marginal PDF of  $\mathbf{Q}$ :

$$\mathbf{f}_{\mathbf{Q}}(q) = \int_{-\infty}^0 \mathbf{f}_{\mathbf{Q},\mathbf{W}}(q, w)dw = \int_{-\infty}^0 -w \cdot \mathbf{f}_{\mathbf{W}}(w)\mathbf{f}_{\mathbf{V}}(wq)dw \quad (8)$$

Combining equations (5) and (8), we have the general case:

$$\mathbf{f}_{\mathbf{Q}}(q) = \int_{-\infty}^{\infty} |w| \cdot \mathbf{f}_{\mathbf{W}}(w)\mathbf{f}_{\mathbf{V}}(wq)dw \quad (9)$$

## 2 The PDF of The $t$ -distribution

If we have a random variable  $\mathbf{T}$  that satisfy:

$$\mathbf{f}_{\mathbf{T}}(t) = \frac{\mathbf{Z}}{\sqrt{\mathbf{U}/\nu}}$$

where  $\mathbf{Z} \sim \mathcal{N}(0, 1)$ ,  $\mathbf{U} \sim \chi^2(\nu)$  and  $\nu > 0$  is the degree of freedom. Then we say  $\mathbf{T}$  follows a  $t$ -distribution of a degree of freedom  $\nu$ :

$$\mathbf{T} \sim \mathcal{T}(\nu)$$

We want to figure out the PDF of  $\mathbf{T}$ :  $\mathbf{f}_{\mathbf{T}}(t)$ .

First, by definition, we have:

$$\mathbf{f}_{\mathbf{Z}}(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \text{ and } \mathbf{f}_{\mathbf{U}}(u) = \frac{1}{\Gamma\left(\frac{\nu}{2}\right) 2^{\frac{\nu}{2}}} u^{\frac{\nu}{2}-1} e^{-\frac{u}{2}}$$

The denominator of the random variable  $\mathbf{T}$  is  $\sqrt{\mathbf{U}/\nu}$ . We let  $\mathbf{X} = \sqrt{\mathbf{U}/\nu}$ . We need to figure out the PDF of  $\mathbf{X}$  first. Using the exact same strategy, we start with:

$$\begin{aligned} \mathbb{F}_{\mathbf{X}}(x) &= \mathbb{P}(\mathbf{X} \leq x) = \mathbb{P}\left(\sqrt{\frac{\mathbf{U}}{\nu}} \leq x\right) = \mathbb{P}\left(\frac{\mathbf{U}}{\nu} \leq x^2\right) \\ &= \mathbb{P}(\mathbf{U} \leq \nu x^2) = \mathbb{F}_{\mathbf{U}}(\nu x^2) \end{aligned}$$

Taking the derivative with respect to  $\mathbf{X}$ , we have:

$$\mathbf{f}_{\mathbf{X}}(x) = 2\nu x \mathbf{f}_{\mathbf{U}}(\nu x^2)$$

Expanding the  $\chi^2$  PDF  $\mathbf{f}_{\mathbf{U}}(\nu x^2)$ , we have:

$$\mathbf{f}_{\mathbf{X}}(x) = 2\nu x \cdot \frac{1}{\Gamma\left(\frac{\nu}{2}\right) 2^{\frac{\nu}{2}}} (\nu x^2)^{\frac{\nu}{2}-1} e^{-\frac{\nu x^2}{2}} = 2\nu x \cdot \frac{1}{\Gamma\left(\frac{\nu}{2}\right)} 2^{-\frac{\nu}{2}} \nu^{\frac{\nu}{2}-1} x^{\nu-2} e^{-\frac{\nu x^2}{2}}$$

Be patient and merge terms of the same colour, we have:

$$\mathbf{f}_{\mathbf{X}}(x) = \frac{2^{1-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \nu^{\frac{\nu}{2}} x^{\nu-1} e^{-\frac{\nu x^2}{2}} \quad (10)$$

Apparently,  $\mathbf{T} = \frac{\mathbf{Z}}{\mathbf{X}}$ . Using equation (9), we have:

$$\mathbf{f}_{\mathbf{T}}(t) = \int_{-\infty}^{\infty} |x| \mathbf{f}_{\mathbf{X}}(x) \mathbf{f}_{\mathbf{Z}}(tx) dx$$

Put equation (10) into the above formula, expand  $f_{\mathbf{Z}}(tx)$  and notice that  $x$  is

non-negative, we have:

$$\mathbb{f}_{\mathbf{T}}(t) = \int_0^\infty x \cdot \frac{2^{1-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right)} \nu^{\frac{\nu}{2}} x^{\nu-1} e^{-\frac{\nu x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2 x^2}{2}} dx$$

Bring the constant terms in front of the integration and merge similar terms, we have:

$$\begin{aligned} \mathbb{f}_{\mathbf{T}}(t) &= \frac{2^{1-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right) \sqrt{2\pi}} \cdot \nu^{\frac{\nu}{2}} \int_0^\infty x \cdot x^{\nu-1} e^{-\frac{\nu x^2}{2}} \cdot e^{-\frac{t^2 x^2}{2}} \\ &= \frac{2^{1-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right) \sqrt{2\pi}} \cdot \nu^{\frac{\nu}{2}} \int_0^\infty x^\nu e^{-\frac{\nu+t^2}{2} x^2} dx \end{aligned} \quad (11)$$

Now we need to figure out the **blue part** in equation (11). It resembles the **Gamma distribution**. The PDF of the Gamma distribution with a shape parameter  $\alpha$  and a scale parameter  $\theta$  is:

$$\frac{1}{\Gamma(\alpha)\theta^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta}}$$

However, we have a  $x^2$  in the exponent of  $e$  in the **blue part** of equation (11), so we need to modify a bit. We let  $x^2 = y$ . Then we have:

$$\begin{aligned} x^\nu &= y^{\frac{\nu}{2}} \\ x^2 = y &\Rightarrow 2x dx = dy \Rightarrow dx = \frac{1}{2x} dy = \frac{1}{2\sqrt{y}} dy = \frac{1}{2} y^{-\frac{1}{2}} dy \end{aligned}$$

Put those into the **blue part** of equation (11), we have:

$$\begin{aligned} \int_0^\infty x^\nu e^{-\frac{\nu+t^2}{2} x^2} dx &= \int_0^\infty y^{\frac{\nu}{2}} e^{-\frac{\nu+t^2}{2} y} \cdot \frac{1}{2} y^{-\frac{1}{2}} dy = \frac{1}{2} \int_0^\infty y^{\frac{\nu-1}{2}} e^{-\frac{\nu+t^2}{2} y} dy \\ &= \frac{1}{2} \int_0^\infty y^{\frac{\nu+1}{2}-1} e^{-\frac{y}{\frac{2}{\nu+t^2}}} dy \end{aligned} \quad (12)$$

Apparently, the above integration looks like a Gamma distribution with a shape parameter  $\alpha = \frac{\nu+1}{2}$  and a scale parameter  $\theta = \frac{2}{\nu+t^2}$ . Therefore, we could rewrite equation (12) as:

$$\int_0^\infty x^\nu e^{-\frac{\nu+t^2}{2} x^2} dx = \frac{\Gamma\left(\frac{\nu+1}{2}\right) \cdot \left(\frac{2}{\nu+t^2}\right)^{\frac{\nu+1}{2}}}{2} \int_0^\infty \frac{1}{\Gamma\left(\frac{\nu+1}{2}\right) \cdot \left(\frac{2}{\nu+t^2}\right)^{\frac{\nu+1}{2}}} y^{\frac{\nu+1}{2}-1} e^{-\frac{y}{\frac{2}{\nu+t^2}}} dy$$

The **red part** is a full model of a Gamma distribution, so its value is 1. Therefore,

we have

$$\int_0^{\infty} x^{\nu} e^{-\frac{\nu+t^2}{2}x^2} dx = \frac{\Gamma\left(\frac{\nu+1}{2}\right) \cdot \left(\frac{2}{\nu+t^2}\right)^{\frac{\nu+1}{2}}}{2}$$

Put it back to equation (11) and rearrange the terms:

$$\begin{aligned} \mathbf{f}_{\mathcal{T}}(t) &= \frac{2^{1-\frac{\nu}{2}}}{\Gamma\left(\frac{\nu}{2}\right) \sqrt{2\pi}} \cdot \nu^{\frac{\nu}{2}} \cdot \frac{\Gamma\left(\frac{\nu+1}{2}\right) \cdot \left(\frac{2}{\nu+t^2}\right)^{\frac{\nu+1}{2}}}{2} \\ &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \cdot \frac{2^{1-\frac{\nu}{2}}}{2\sqrt{2\pi}} \cdot \nu^{\frac{\nu}{2}} \cdot \left(\frac{2}{\nu+t^2}\right)^{\frac{\nu+1}{2}} \\ &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \cdot \frac{2^{-\frac{\nu+1}{2}}}{\sqrt{\pi}} \cdot \nu^{-\frac{1}{2}} \cdot \nu^{\frac{\nu+1}{2}} \cdot \left(\frac{2}{\nu+t^2}\right)^{\frac{\nu+1}{2}} \\ &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)} \cdot \frac{2^{-\frac{\nu+1}{2}}}{\sqrt{\pi\nu}} \cdot \frac{2^{\frac{\nu+1}{2}} \cdot \nu^{\frac{\nu+1}{2}}}{(\nu+t^2)^{\frac{\nu+1}{2}}} \\ &= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right) \sqrt{\pi\nu}} \cdot \left(\frac{\nu}{\nu+t^2}\right)^{\frac{\nu+1}{2}} \end{aligned} \tag{13}$$

That's the PDF of  $\mathcal{T}(\nu)$ . Typically, it is often written as:

$$\mathbf{f}_{\mathcal{T}}(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right) \sqrt{\pi\nu}} \cdot \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}$$

### 3 The PDF of The $\mathcal{F}$ -distribution

If we have a random variable  $\mathbf{X}$  that satisfy:

$$\mathbf{f}_{\mathbf{X}}(x) = \frac{\mathbf{U}_1/n}{\mathbf{U}_2/m}$$

where  $\mathbf{U}_1 \sim \chi^2(n)$ ,  $\mathbf{U}_2 \sim \chi^2(m)$  and  $n, m > 0$  are the degrees of freedom of  $\mathbf{U}_1$  and  $\mathbf{U}_2$ , respectively. Then we say  $\mathbf{X}$  follows an  $\mathbf{F}$ -distribution of a degree of freedom  $n$  in the numerator and  $m$  in the denominator:

$$\mathbf{X} \sim \mathcal{F}(n, m)$$

We want to figure out the PDF of  $\mathbf{X}$ :  $\mathbf{f}_{\mathbf{X}}(x)$ .

Since  $n$  and  $m$  are just some numbers, let's first use equation (9) to derive the PDF of  $\frac{U_1}{U_2}$ . Let  $Y = \frac{U_1}{U_2}$ . Using equation (9), we have:

$$f_Y(y) = \int_{-\infty}^{\infty} |u_2| f_{U_2}(u_2) f_{U_1}(yu_2) du_2$$

Note both  $u_1$  and  $u_2$  are non-negative and put the values into the  $\chi^2$  PDFs:

$$\begin{aligned} f_Y(y) &= \int_0^{\infty} u_2 \cdot \frac{1}{\Gamma\left(\frac{m}{2}\right) 2^{\frac{m}{2}}} u_2^{\frac{m}{2}-1} e^{-\frac{u_2}{2}} \cdot \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} (yu_2)^{\frac{n}{2}-1} e^{-\frac{yu_2}{2}} du_2 \\ &= \frac{y^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m}{2}\right) 2^{\frac{n+m}{2}}} \int_0^{\infty} u_2 \cdot u_2^{\frac{m}{2}-1} \cdot u_2^{\frac{n}{2}-1} \cdot e^{-\frac{u_2}{2}} \cdot e^{-\frac{yu_2}{2}} du_2 \\ &= \frac{y^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m}{2}\right) 2^{\frac{n+m}{2}}} \int_0^{\infty} u_2^{\frac{n+m}{2}-1} e^{-\frac{y+1}{2}u_2} du_2 \end{aligned} \quad (14)$$

Similar to the previous examples, the blue part of equation (14) looks like a Gamma distribution. We re-write it as:

$$\begin{aligned} f_Y(y) &= \frac{y^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m}{2}\right) 2^{\frac{n+m}{2}}} \int_0^{\infty} u_2^{\frac{n+m}{2}-1} e^{-\frac{u_2}{y+1}} du_2 \\ &= \frac{y^{\frac{n}{2}-1} \cdot \Gamma\left(\frac{n+m}{2}\right) \cdot \left(\frac{2}{y+1}\right)^{\frac{n+m}{2}}}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m}{2}\right) 2^{\frac{n+m}{2}}} \int_0^{\infty} \frac{1}{\Gamma\left(\frac{n+m}{2}\right) \cdot \left(\frac{2}{y+1}\right)^{\frac{n+m}{2}}} u_2^{\frac{n+m}{2}-1} e^{-\frac{u_2}{y+1}} du_2 \end{aligned} \quad (15)$$

The blue part of equation (15) is a full model of a Gamma distribution with the shape parameter  $\alpha = \frac{n+m}{2}$  and a scale parameter of  $\theta = \frac{2}{y+1}$ . It is equal to 1. Therefore, we have:

$$f_Y(y) = \frac{y^{\frac{n}{2}-1} \cdot \Gamma\left(\frac{n+m}{2}\right) \cdot \left(\frac{2}{y+1}\right)^{\frac{n+m}{2}}}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m}{2}\right) 2^{\frac{n+m}{2}}} = \frac{\Gamma\left(\frac{n+m}{2}\right) y^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m}{2}\right) (y+1)^{\frac{n+m}{2}}} \quad (16)$$

Now we need to figure out the PDF of  $X = \frac{m}{n} Y$ , which is straightforward based on **Section 5.1** from the **Extra Reading Material** from **Lecture 13**. Put into

equation (16), we have:

$$\begin{aligned}
 f_{\mathbf{X}}(x) &= \frac{1}{\frac{m}{n}} \cdot f_{\mathbf{Y}}\left(\frac{x}{\frac{m}{n}}\right) = \frac{n}{m} \cdot f_{\mathbf{Y}}\left(\frac{n}{m}x\right) \\
 &= \frac{n}{m} \cdot \frac{\Gamma\left(\frac{n+m}{2}\right) \left(\frac{n}{m}x\right)^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m}{2}\right) \left(\frac{n}{m}x + 1\right)^{\frac{n+m}{2}}} \\
 &= \frac{n}{m} \cdot \frac{\Gamma\left(\frac{n+m}{2}\right) \left(\frac{n}{m}\right)^{\frac{n}{2}-1} x^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m}{2}\right) \left(\frac{n}{m}x + 1\right)^{\frac{n+m}{2}}} \\
 &= \frac{\Gamma\left(\frac{n+m}{2}\right) \left(\frac{n}{m}\right)^{\frac{n}{2}-1} x^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{m}{2}\right) \left(\frac{n}{m}x + 1\right)^{\frac{n+m}{2}}}
 \end{aligned}$$