Lecture 8 Independent Events

BIO210 Biostatistics

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Practice

Question 1: flipping a fair coin three times.

$\mathbb{P}(\{HTT\}) = ?$

$\mathbb{P}(\{\text{two heads}\}) = ?$

Question 2: It is known that there are 0.5% people in general that carry the virus V. A company has a kit to detect this specific virus. The sensitivity and the specificity of the kits are 50% and 50%, respectively. A random person from the population get chosen and tested by the kit. It turns out that the test result is positive. Now what is the probability that the person carries the virus?

Independence of Two Events

Definition 1

Events A and B are independent if $\mathbb{P}\left(B|A\right) = \mathbb{P}\left(B\right), \mathbb{P}\left(A\right) \neq 0$

Meaning of Definition 1: the occurrence of A provides no information about the occurrence of B.

Definition 2

Events A and B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$

Advantages of Definition 2:

- Symmetric with respect to A and B.
- $\mathbb{P}(A)$ or $\mathbb{P}(B)$ can be 0

Independence

Experiment (Lecture 4): keep flipping a coin until we obtain a head for the first time and stop. Let n be the number of flips.

Sample space: $\Omega = \{$H, TH, TTH, TTTH, ...$\}$

$$\mathbb{P}(n) = \frac{1}{2^n},\ n = 1, 2, 3, 4, \cdots$$

Probabilistic model:

$$\mathbb{P}(H) = p$$

$$\mathbb{P}(T) = 1 - p$$

$$\mathbb{P}(TH) = (1-p)p$$

$$\mathbb{P}(TTH) = (1-p)(1-p)p$$

$$\vdots$$

$$\mathbb{P}\left(\underbrace{TTT...TTT}_{n-1 \text{ tails}} H\right) = (1-p)^{n-1}p$$

Independent of a collection of events

Intuitive definition

Information on some of the events does not provide any information about probabilities of the remaining events:

$$\mathbb{P}[(A \cap B \cap C \cap D)|(E \cap F)] = \mathbb{P}\left(A \cap B \cap C \cap D\right)$$

Independent of a collection of events

Mathematics definition

Events $A_1, A_2, A_3, ..., A_n$ are called independent if and only if:

$$\mathbb{P}\left(A_i \cap A_j \cap ... \cap A_q\right) = \mathbb{P}\left(A_i\right) \cdot \mathbb{P}\left(A_j\right) \cdots \mathbb{P}\left(A_q\right)$$

for any distinct indices $i, j, ..., q$ chosen from $\{1, 2, ..., n\}$

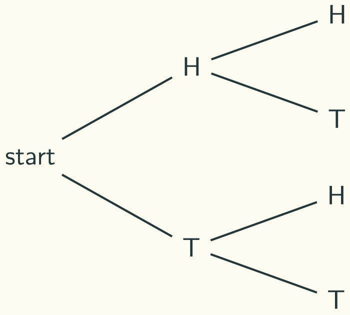
Independent of a collection of events

According to the definition, for a collection of events $\{A_1, A_2, A_3\}$ to be independent, they need to satisfy all the following conditions:

- $\mathbb{P}(A_1 \cap A_2 \cap A_3) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2) \cdot \mathbb{P}(A_3)$
- Pairwise independence:
  - $\mathbb{P}(A_1 \cap A_2) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_2)$
  - $\mathbb{P}(A_1 \cap A_3) = \mathbb{P}(A_1) \cdot \mathbb{P}(A_3)$
  - $\mathbb{P}(A_2 \cap A_3) = \mathbb{P}(A_2) \cdot \mathbb{P}(A_3)$

Independent of a collection of events

Example 1: two independent coin (fair) flips.



$A = \{\text{the first is H}\}$

$B = \{\text{the second is H}\}$

$C = \{\text{the first and the second give the same result}\}$

$\mathbb{P}(A \cap B) = ?$

$\mathbb{P}(A \cap C) = ?$

$\mathbb{P}(B \cap C) = ?$

$\mathbb{P}(C | A \cap B) = ?$

Pairwise independence does not imply independence.

Independent of a collection of events

Example 2: flipping a fair coin 4 times

Sample Space $\Omega = \{$HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, THHT, THTH, TTHH, HTTH, TTTH, TTHT, THTT, HTTT, TTTT$\}$

A=$\{$ HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, THHT $\}$

B=$\{$ THHT, THTH, TTHH, HTTH, TTTH, TTHT, THTT, HTTT $\}$

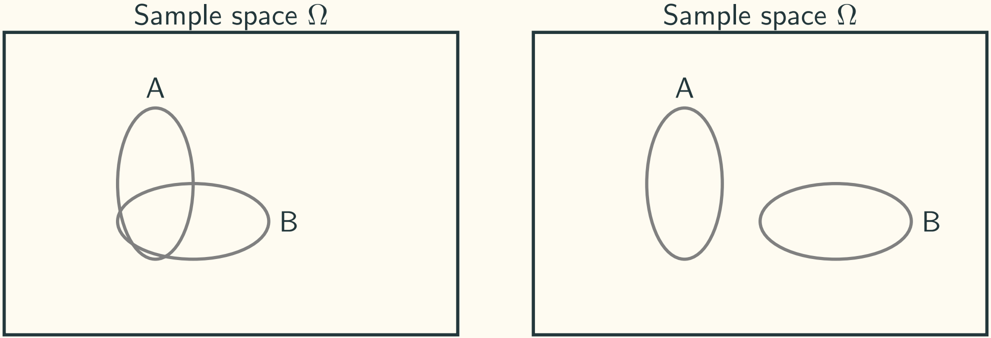
C=$\{$ THHT, THTH, TTHH, HTTH $\}$

$\mathbb{P}(A \cap B \cap C) = ?$

$\mathbb{P}(B \cap C) = ?$

Simple multiplication does not imply independence.

Independent Events

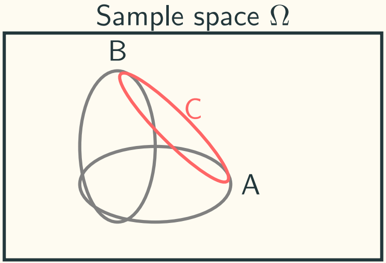


- Venn diagram is not sufficient to display independent events.
- Do not confuse independent events with disjoint events.

Definition

$$\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C) \cdot \mathbb{P}(B|C)$$

Events A and B are independent in the following Venn diagram:



Having independence in the original model does not imply independence in the conditional model.

Conditional independence

Example of conditional independence - a virus detection kit:

- If a person carries the virus, the kit has 90% of the chance of showing a positive result.
- If a person does not carry the virus, the kit has 90% of the chance of showing a negative result.
- The virus is common and non-harmful. In general, 50% of the whole population carry the virus without any symptoms or illness.
- We have a random person called Li Lei. He gets tested by the kit repeatedly. Let event A = { the 4th test is positive } and event B = { the first 3 tests are all positive }

Questions: are events A and B independent? Does the answer depend on if we know Li Lei carries the virus or not?

Conditional independence

- A = { the 4th test is positive }
- B = { the first 3 tests are all positive }

- We know Li Lei carries the virus.
  - $\mathbb{P}(A) = ?$
  - $\mathbb{P}(A|B) = ?$
- We know Li Lei does NOT carry the virus.
  - $\mathbb{P}(A) = ?$
  - $\mathbb{P}(A|B) = ?$
- We don't know if Li Lei carries the virus or not.
  - $\mathbb{P}(A) = ?$
  - $\mathbb{P}(A|B) = ?$

Having independence in the conditional model does not imply independence in the original model.

Independent Events

The Gambler's Fallacy

The Gambler's Fallacy in RPG Games

The Sally Clark Case

- Sudden infant death syndrome (SIDS) is the sudden unexplained death of a child of less than one year of age.
- Clark's first son died in December 1996 within a few weeks of his birth.
- Her second son died in similar circumstances in January 1998.
- She was convicted in November 1999. The convictions were overturned in January 2003.
- As a result of her case, the Attorney-General ordered a review of hundreds of other cases, and two other women had their convictions overturned.

The CESDI Report

Groups	SIDS incidence in this group
Overall population	363 in 472,823
Anybody smokes in the household	1 in 737
Nobody smokes in the household	1 in 5041
No waged income in the household	1 in 486
At least one waged income in the household	1 in 2,088
Mother $<$ 27 years and parity	1 in 567
Mother $>$ 26 years and parity	1 in 1882
None of these factors	1 in 8,543
One of these factors	1 in 1,616
Two of these factors	1 in 596
All three of these factors	1 in 214

The Sally Clark Case

Professor Sir Roy Meadow, a highly respected expert in field of child abuse at the time of the trial:

"you have to multiply 1 in 8,543 times 1 in 8,543 and I think it gives that in the penultimate paragraph. It points out that it's approximately a chance of 1 in 73 million
"

The Sally Clark Case: one of the great miscarriages of justice in modern British legal history